Estimating the Lifetime Distribution of Options by a Numerical Approach

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Abstract: Expected lifetime is an important value for investors or risk managers in American and most exotic options because the contractual expiry is only the maximum expiry where the contract can end. Indeed, if in the case of a vanilla option, the lifetime is equal to the maturity, because the contract may only be exercised at his contractual expiry, that is not the case for American (and most exotic) options since they may be exercised earlier than their maximum expiry. In this article, a numerical method is provided, to value the lifetime distribution of American and most common exotic options. As the approach is based on a tree, volatility smile, dividends and several underlying assets can easily be taken into account. As a matter of fact, the given explanations are for option on index, shares, exchange rate. But exactly the same algorithm can be applied on interest rate trees.

Keywords: American, Exotic, Lifetime, Option, Tree.

Estimating the Lifetime Distribution of Options by a Numerical Approach

One of the most important concepts for investors or risk managers is the lifetime distribution of options. This information can be used in several ways either in the hedging of an option or in a speculative strategy. Hereafter, only two examples are mentioned, among all, in order to present how one can deal with the expected lifetime of an option. The first example is based on the buyer of an American put option, expiring (*at the most*) in one year, who wants to insure a given portfolio. Then, the owner of the put is not insured during the whole period, but until the exercise occurs. Going on insuring his portfolio will probably conduce him into buying another put option. Therefore, investing in an American put, whose expected lifetime is equal to one year, would certainly be more efficient. The second proposed example is typical of the volatility trading technique. This time, we consider a trader who is selling an American straddle, that is selling simultaneously an American call and an American put with same strike and expiry. This short position has not the same lifetime depending on whether the underlying price rises or falls. If it falls down, then the call is worth nothing and the put will probably be exercised. Whereas if the price rises, the American put is worth nothing but the American call would probably not be exercised and therefore the position can go on. So, the sensitivity of such an exposure can be high enough in the down way to make him prefer a strangle with an out of the money put. In order to choose the strike of the put, he could appreciate to do it, knowing the attached risk, i.e. the true expected lifetime. For exotic options like reversal barrier ones[1], the lifetime distribution is an even more essential parameter for a clear understanding of the underlying risk of the contract.

In order to provide the lifetime law of an American option, one needs first to estimate the exercise frontier. That is, the value of the underlying asset, depending on the time under (or above) which the option is always exercised. Because the option is exercised only if the underlying asset crosses this frontier, the knowledge of the frontier through time is required in order to estimate the lifetime distribution of an American option. A great deal of authors studied this frontier in a continuous time setting (see for example Chesney & Lefoll [4]). With the use of PDE or tree it is straightforward to estimate this frontier in a discrete time setting and by the way of interpolation to provide a correct approximation.

Recently, Douady [9] simultaneously studied the pricing of some exotic options and their lifetime distribution. This was done in a continuous time setting and only for options depending on one risky asset. The study had been led under the hypothesis of a geometric Brownian motion with constant parameters and with the assumption that no cash dividend could be paid to the owners of the underlying assets.

The following method that we suggest can go beyond all the restrictions due to the difficulties encountered in using a continuous time setting, with no high cost of calculation time, since it is based on a tree. For simplicity, the presentation of the algorithm is restricted in the case of the well known Cox, Ross and Rubinstein [6] (next CRR) tree. Obviously, other tree models (like [5], [8], [10], [14]) may be used in order to model other assumptions as it has been done in the following for rainbow options where the quadrinomial tree of Augros and Moreno [1] has been employed. So on for interest rate trees if one is interested in the lifetime distribution of an interest rate option.

The article is organized in two parts. In the first one, the algorithm providing the lifetime distribution is exposed whereas in the second one some results are discussed.

I. Estimation of the lifetime distribution of options

Douady, referring to the simple definition of the duration concerning contracts contingent to interest rate, used the term of duration to define the expected value of the lifetime of an option. According to his work, the same notation will be used hereafter.

The estimation of the lifetime distribution of options needs several steps in order to calculate the probability of crossing the exercise frontier at a fixed date. Once this result is obtained, the lifetime distribution of the option can be straightforwardly assessed. The following discussion exposed first the algorithm leading to the expected results in the case of American options. Extensions to exotic options will be presented further.

A. Implementing the method

The suggested numerical approach is constructed from the binomial tree of CRR. All extensions of this tree, published since their research work, can be directly integrated into our algorithm. But, for simplicity the discussion is voluntary restricted around the initial binomial model.

We quickly remind that into this discrete model[2], the time to maturity t of the option is decomposed into N periods of the same length $Dt = \frac{t}{N}$. During each time step, the price of the risky asset either rises or falls. In the up state, the value of the underlying has been multiplied by $u = e^{s\Delta t}$ whereas in the down state, the price has been multiplied by $d = \frac{1}{u}$. In the risk neutral world, the probability p of reaching the up state is worth : $p = \frac{e^{r\Delta t} - d}{u - d}$.

During the backward recursion required to value the option, the exercise frontier is spotted. More specifically, the region where the anticipated exercise appears is marked in another tree with the value 1. Elsewhere, values in the tree are set to 0. Therefore, one possible example could look like this one:

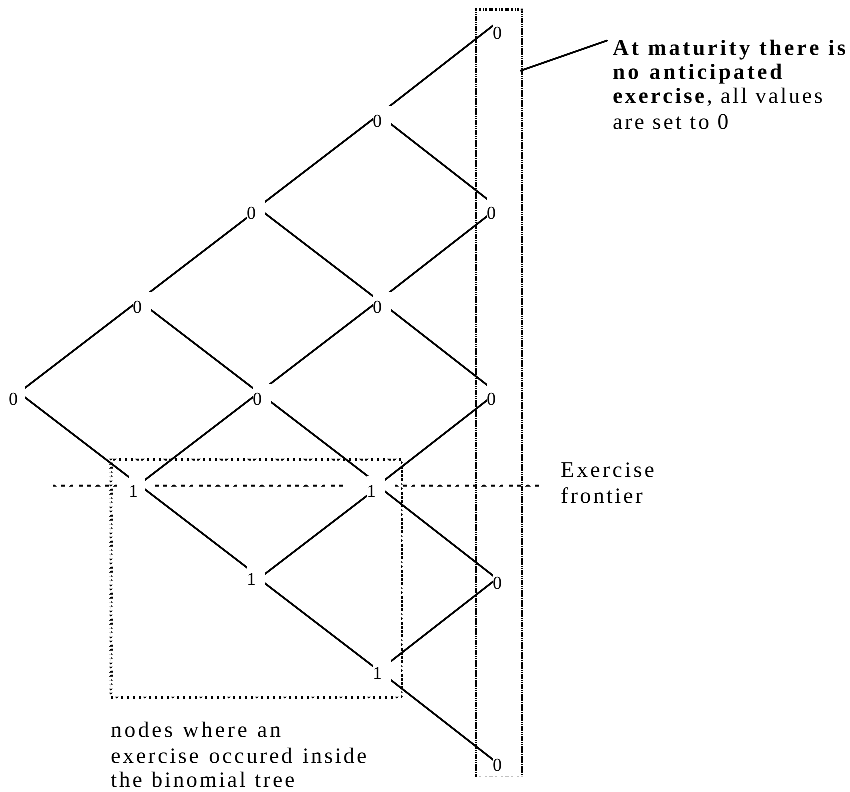


figure 1

Because the probability of crossing the exercise frontier has to be estimated in the real world, the binomial process of S has now to be diffused inside the historical world. That means that this process: $\frac{dS}{S} = m dt + s\, dZ$ instead of this one $\frac{dS}{S} = r dt + s\, dZ$ has to be discretisized. By doing so, the values of S inside the binomial tree don't change. But, the probability of going upward is transformed to become $q = \frac{e^{m\Delta t} - d}{u - d}$. Consequently there is no need to diffuse the value of the risky asset in the historical world because only the probabilities can be changed.

Because the key of the calculation of the lifetime distribution rests on the estimation of the future value of *m*, a complete study of the law of *m* should be computed. Indeed, the sensitivity of the duration of an option may be high in reference to the value of *m*.

In order to simplify the presentation, the lifetime distribution is not estimated in the real world but in the risk neutral one. This really doesn't change the global property of the method but only the shape (or moments) of the distribution.

Before introducing the algorithm, a simple example is given thanks to a three-period[3] binomial tree. The tree containing the information of anticipated exercise is supposed to be the next one:

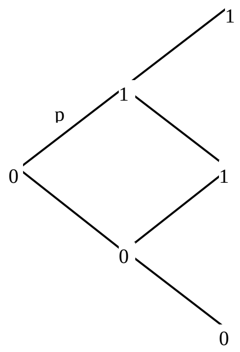


figure 2

Starting from the initial date, the probability of reaching a node where an exercise occurs for the first time can be assessed thanks to a forward induction of the tree. Here, these probabilities are:

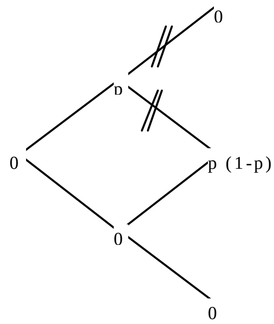
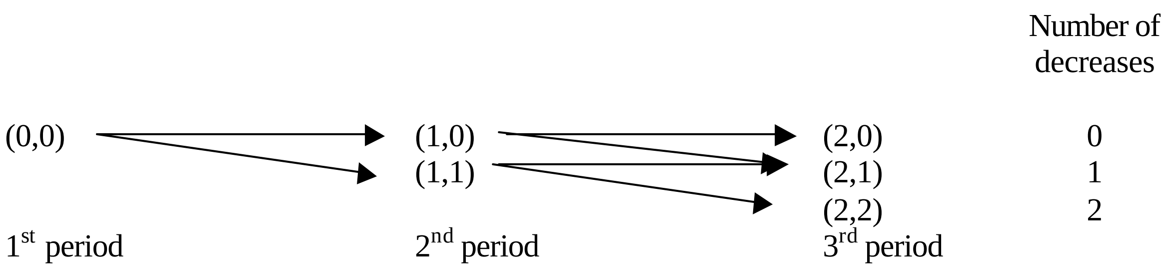


figure 3

All the next trajectories of S in which the first move has been upward must not be taken into account in the calculation of the wanted probabilities because the exercise has been released at the end of the first move and therefore the option no longer exists.

The algorithm to solve this estimation problem can be decomposed in a two-step process given hereafter. Since the binomial model is used, the values of the wanted probabilities are stored in a triangular matrix named EndProb(.,.). Two other triangular matrices P(.,.) and Prob(.,.) are needed to value the lifetime distribution. For all of these, the point (n,i) refers to the following : n, is the number of periods occurred and i, the number of decreases experienced by the risky asset. The first two time steps are developed below:



For a clearer understanding of the algorithm, the probability p has been noted UpProb whereas the probability (1-p) has been noted DownProb. Prob(.,.) contains either the value 1 or 0 according to whether or not the node is stated in the exercise area (more generally above or under a specified frontier), as shown in figure 1. The values of UpProb and DownProb could of course be time and price depending. The first step of the algorithm is now introduced :

```
Step 1 :    Calculation of P(.,.)
Set P(0,0) To 1
From Period=1 To N Do
        If Prob(Period,1) = 1 Then
              P(Period,1) = 0
        Else
              P(Period,1) = P(Period-1,1) * UpProb

        From Node = 1 To Period-1 Do
              If Prob(Period,Node) = 1 Then
                   P(Period,Node) = 0
              Else
                   P(Period,Node) = P(Period-1,Node) * UpProb
                                   + Period-1,Node-1) * DownProb

        If Prob(Period,Period) = 1 Then
              P(Period,Period) = 0
        Else
              P(Period,Period) = P(Period-1,Period-1) * DownProb
```

Once this matrix calculated, the second step of the algorithm can be proceeded :

```
Step 2 :    Calculation of EndProb

We suppose there is no immediate exercise

Set EndProb To 0
From Period=0 To N-2 Do
     From Node=0 To Period-1 Do
        If Prob(Period,Node) = 0 Then
           If Prob(Period+1,Node) = 1 Then
              If EndProb (Period,Node) > 0 Then
                 EndProb (Period+1,Node) =  EndProb(Period+1,Node)
                                            +  (1- EndProb (Period,Node)) * UpProb
              Else
                 EndProb (Period+1,Node) = EndProb (Period+1,Node)
                                            +  P(Period,Node) * UpProb

           If Prob(Period+1,Node+1) = 1 Then
              If EndProb(Period,Node) > 0 Then
                 EndProb(Period+1,Node+1) = EndProb(Period+1,Node+1)
                                            + (1- EndProb(Period,Node)) * DownProb
              Else
                 EndProb(Period+1,Node+1) = EndProb(Period+1,Node+1)
                                            +  P(Period,Node) * DownProb
```

B. Extracting the lifetime distribution from the EndProb matrix

The algorithm exposed above immediately leads to the knowledge of the lifetime distribution. Indeed, the calculated EndProb matrix contains the wanted data. The statement is illustrated in the case of an American put with a seven time step tree for which the EndProb matrix has been written below:

						0.00
					0.00	
				0.00		0.00
			0.00		0.00	
		0.00		0.00		0.00
	0.00		0.00		0.00	
0.00		0.00		0.00		0.00
	0.00		0.00		0.00	
		0.00		0.00		0.13
			0.11		0.09	
				0.00		0.00
					0.00	
						0.00

Period : 1 2 3 4 5 6

With: $S = 100$; $K = 100$; $r = 5\%$; $t = 1$ year ; $s = 25\ \%$.

In that case, the lifetime distribution of the option, in the risk-neutral world and according to the model employed, is:

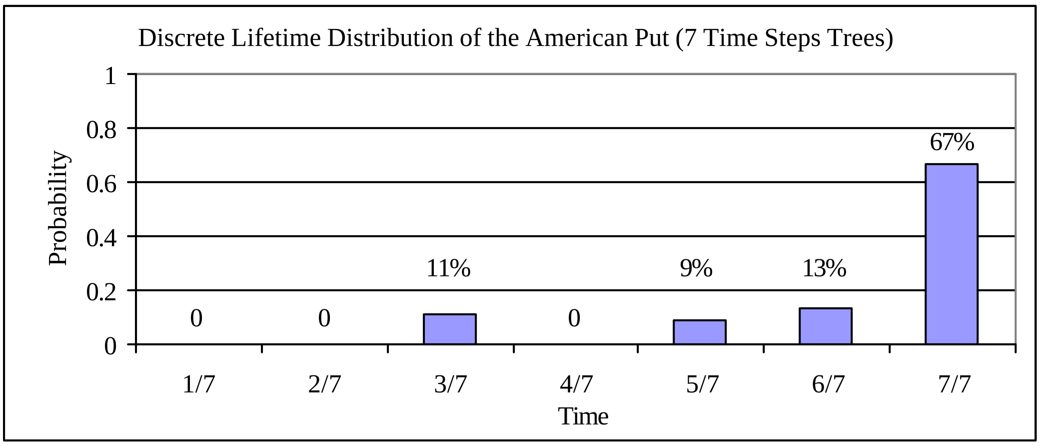


figure 4

As a matter of fact, the value that the option extinguishes at a given period is worth the sum *SEP* of all the values contained in the EndProb matrix at the same period[4]:

$$SEP(i) = \sum_{j=0}^{i} EndProb(i, j) .$$

Using more time steps[5], the lifetime distribution is taking shape:

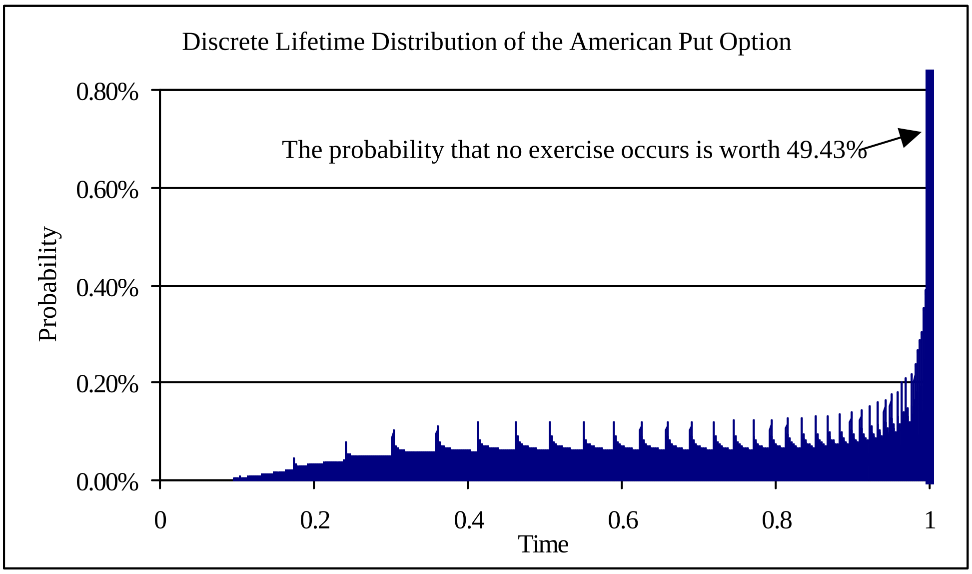


figure 5

From this result, it is straightforward to estimate the mean of the lifetime of the option, i.e. the duration:

$$Duration = Maturity \times \left(1 - \sum_{i=0}^{N-1} SEP(i)\right) + \sum_{i=0}^{N-1} \left(t_i SEP(i)\right)$$

The algorithm is convergent because the lifetime distribution is estimated from the probability law that the asset price crosses a deterministic frontier. Now, the binomial model of CRR here used ensures the convergence of this (discrete) probability law toward the one expressed in a continuous time setting. This definitely ensures the convergence of the method.

For the most exotic option, the extension is really direct. Firstly, whatever the payoff structure, the algorithm needs no transformation. Secondly, it is straight to integrate an out barrier, since this barrier can be considered as another anticipated exercise frontier. Lastly, the algorithm can be extended to options contingent to several underlying or others stochastic process. All these extensions are straightforward enough for not entering into further details of the necessary adjustments. Only, the principal results are discussed hereafter.

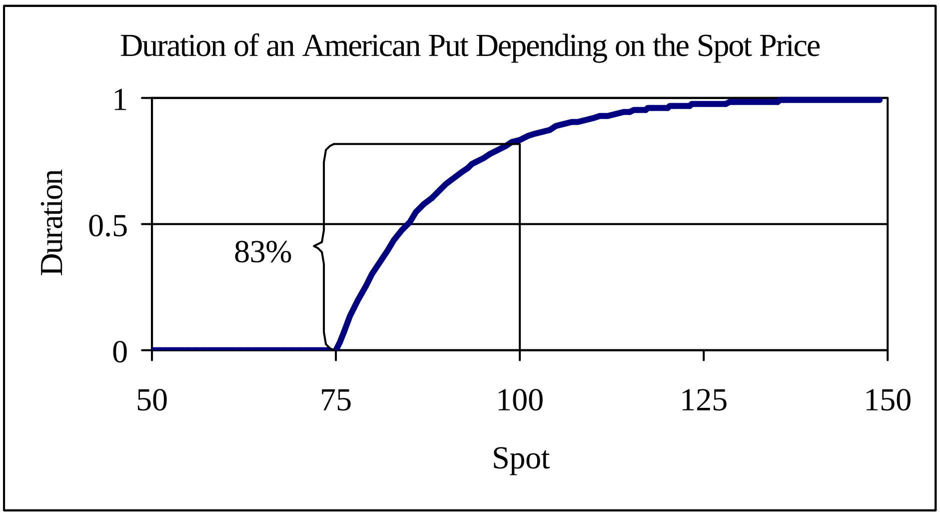
II. Results

There are so many different options and cases that only some main results are studied in this section. The large amount of collected information shows the impact in the lifetime of several aspects like the payment date of a dividend, the value of the spot and for 2-color rainbow options, the impact of the correlation between the returns of two risky assets.

The results for American options are first introduced. Then the study is lead with an up & out call and with rainbow options.

A. American Put Option

American put options can be exercised very prematurely, same during their issue day. The duration of an American put option has been valued in function of the spot. The found sensibility of the duration is high like the following graphic reveals:



with S = . ; K = 100 ; r = 5 % ; s = 25 % ; t = 1 year.

figure 6

When the spot is worth less than 75, the option is immediately exercised. As a result, the duration is worth 0. When the spot is definitely up the strike then the probability that an anticipated exercise occurs tends to 0. The drawn inference is that the duration tends towards 1. Between these two ends, the graphic reveals that the duration highly differs when the option is in the money. A move of 25 % in the spot (100 downward 75) implies a fall of 100 % of the expected lifetime (83% downward 0).

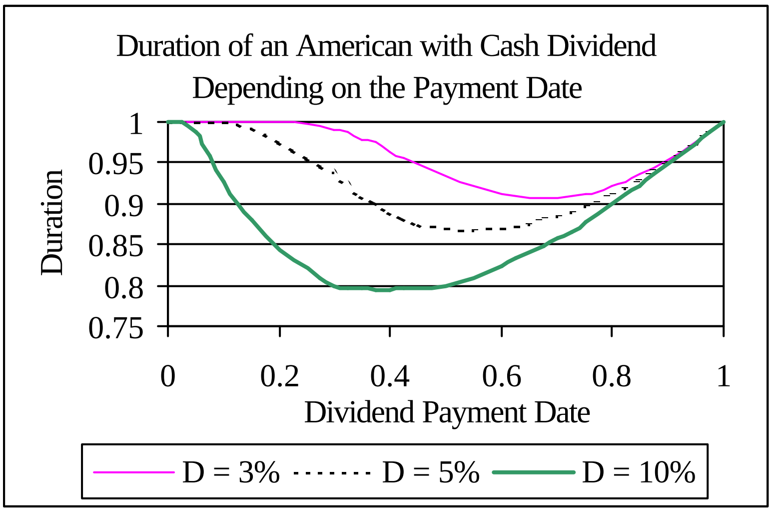
B. American Call Option with a Cash Dividend

It can be proven that if there is no payable dividend during the life of an American call, then exercising it prematurely is not efficient (in the Black Scholes world[6]). As a matter of fact, Roll [13] proved in 1977 that the following condition was necessary for the occurrence of an anticipated exercise of an American call:

$$D > \frac{r'K}{1+r'}$$

where D is the amount of the dividend, and r' the value of the forward risk free rate starting from the date of the payment of the dividend and ending at the expiry of the call. So in order to find a duration different to the expiry, a cash paid dividend has been introduced to study the duration.

The duration of an American call has been calculated with three different amounts of dividend in function of their payment date. The results are given in the following graphic:

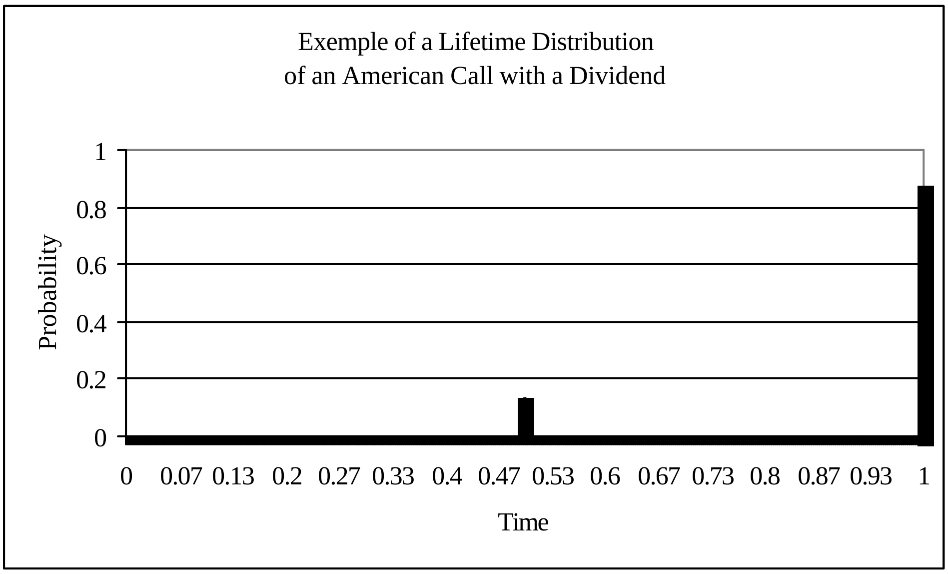


with S = 100 ; K = 100 ; r = 5 % ; s = 25 % ; t = 1 year.

figure 7

If the payment of the dividend occurs at the very beginning of the first day, then it is not efficient to exercise the option. If the payment date is on the last day then the option won't be exercised until this day. These two points accounted for the reason why the two ends of each curves are worth 1.

Unlike the case of an American put option, there are only two dates during which the American call can be exercised. So, the misunderstanding between the duration and the payment date of the dividend where an anticipated exercise can happen should not be done. There is certainly no exercise possibility at the date of the duration. As a matter of fact, in the case here exposed, a trader will probably not use the duration in order to directly hedge his exposure. But he may find more appropriate to use the lifetime distribution of the option, which is here:



with S = 100 ; K = 100 ; r = 5 % ; s = 25 % ; t = 1 year ; D = 3 % paid in 6 months.

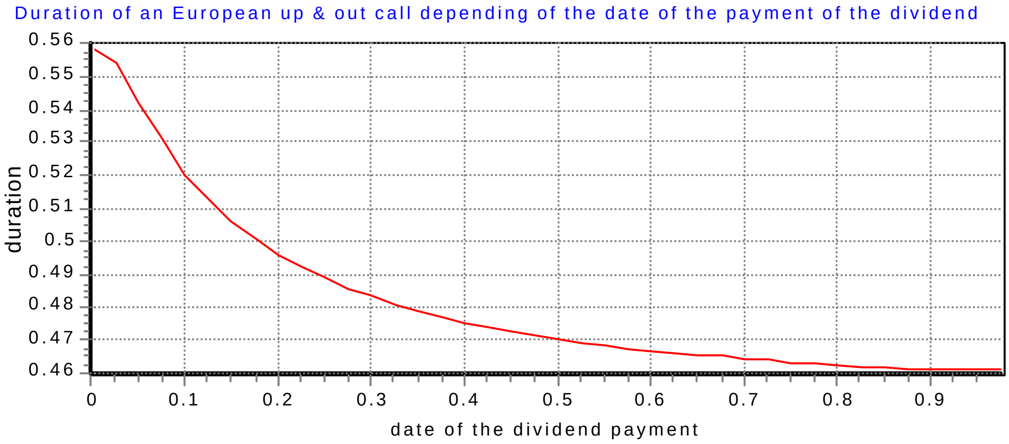
figure 8

To clarify the reason why the use of the lifetime distribution may be preferred according to the duration, an extreme example is developed. If the payment date of the dividend succeeds the issue date by one day, then the duration may be worth nearly 0.5 year and the lifetime distribution may be uniformly shared between the payment day of the dividend and the expiry of the option. In such a case, a trader will probably consider separately the two different dates of exercise, respect to their probability, instead of building a hedging strategy for the 0.5 year expected lifetime.

C. European Up & Out Call

Since there is no need in calculation of the exercise frontier, it is easier and quicker to compute the algorithm in the case of an European up & out call. In fact, there is no real difficulty to deal neither with an American up & out call nor with shark option (barrier option with an included positive rebate).

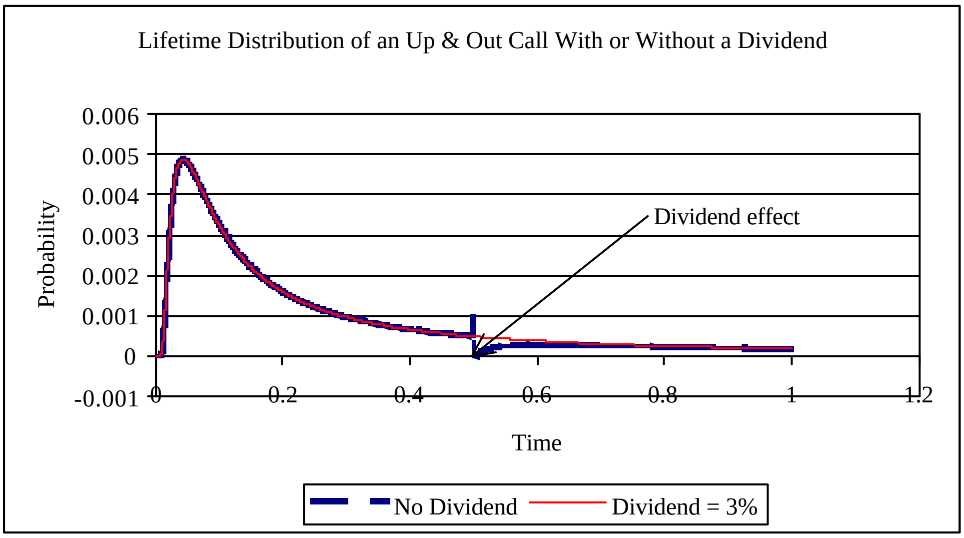
Hereafter, the duration has been valued in function of the date of the dividend payment. Depending on this payment date, the duration falls down from 0.56 to 0.46, that is a close 18 % drop as the following graphic shows:



with S = 100 ; B = 110 ; K = 100 ; r = 5 % ; s = 25 % ; t = 1 year ; D = 3 % paid in X months.

figure 9

The next two curves give an insight into the effect of the paid dividend on the law shape (a continuous time analysis, with no cash dividend, of this value is furnished for example by both Dana & Jeanblanc-Picqué [8] and Rich[12]):



with S = 100 ; B = 110 ; K = 100 ; r = 5 % ; s = 25 % ; t = 1 year ; D = 3 % paid in X months.

(To provide visible distribution shapes, the probabilities that no anticipated exercise occurs have been removed)

figure 10

Most of the time, the two curves are superimposed. But, at the dividend payment date, they are clearly different. According to financial theory, when a dividend is paid, the stock price drops. As the barrier is upward, the effect of the dividend lowers the probability of reaching the barrier.

Using the trinomial Cheuk & Vorst's [5] model would have been more efficient because the binomial CRR's tree is not really adapted to price barrier option[7]. The following graphic shows the error encountered using the CRR's tree :

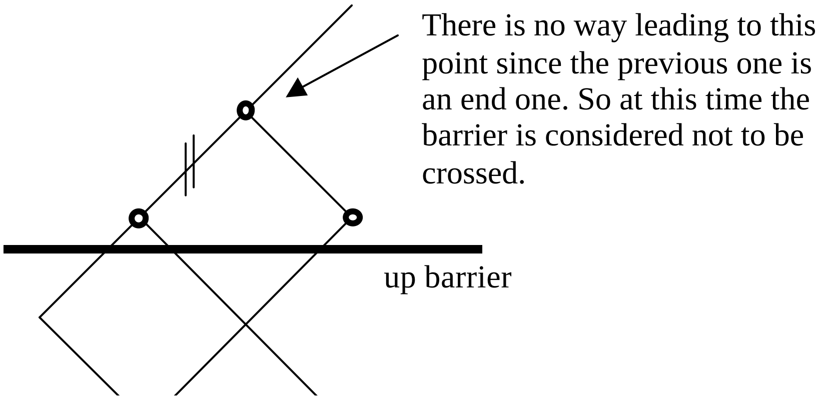


figure 11

Using a binomial scheme may conduce to have a probability of reaching the barrier at a certain date worth zero. If instead we used the adapted Cheuk & Vorst's trinomial scheme, the probabilities are more accurate[8] :

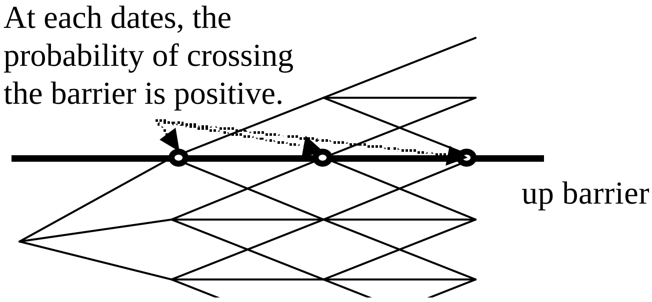


figure 12

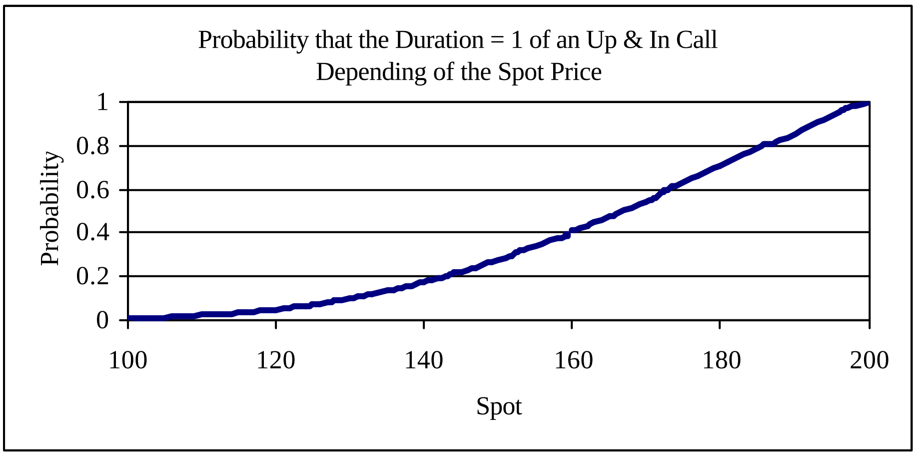
As a matter of fact, exactly the same problem occurs with American options since one is looking for the life time of a crossing of a frontier. By the same way, one can take advantage of the flexibility of the Cheuk & Vorst's tree to solve this difficulty.

D. European Up & In Call

The lifetime of an up & in call is worth the maturity of the option whether or not the in barrier is crossed. Hence there is no reason to study directly the lifetime of an in barrier option. But, on the other hand, the call is activated only if the spot passes through the barrier. At this time the remaining lifetime of the option is equal to t-t_B (with $t_B < t$, the date at which the spot jumps over the barrier). But from the starting time t_0, this means that the lifetime of the call is worth t. Instead if the spot never cross the barrier, one can consider that the lifetime of the call was null.

Using the algorithm in the same manner than for the out barrier option, the $SEP(i)$ values are established. Remind that $SEP(i)$ is worth the probability of crossing the in barrier at time t_i, knowing it never occurred before. This means that with probability $\sum_i SEP(i)$ the option has a concrete lifetime worth t and with probability $1-\sum_i SEP(i)$ the option has a real lifetime worth 0.

The next graphics shows the probability that the "concrete" duration referring to the initial spot price of the risky asset is worth one :



with B = 200 ; K = 100 ; r = 5 % ; s = 25 % ; t = 1 year.

figure 13

As expected, the probability is growing up as the initial spot price comes nearer to the in barrier.

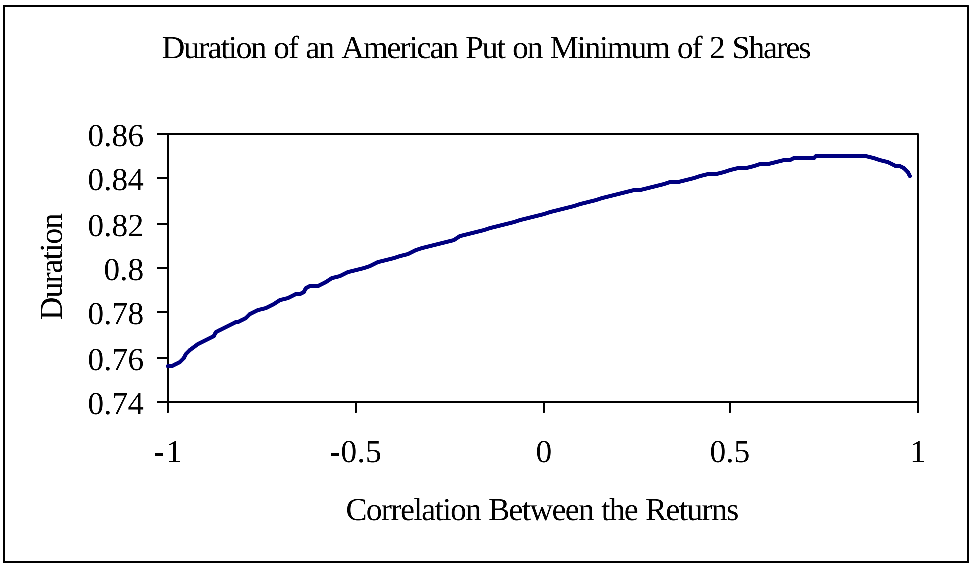
E. Rainbow Options

The algorithm, introduced in the first section, can easily be extended to options depending on several assets[9]. The chosen example is made with an American two-colors rainbow put on minimum. The pay off of such an option is:

$$payoff = Max\{K - Min(S_1(t), S_2(t)); 0\}$$

where $S_1(t)$ and $S_2(t)$ are the prices of the two risky assets at the expiry of the option.

The next graphic shows the sensibility of the duration respect to the correlation between the returns of the two risky assets:



with $S_1 = S_2 = 100$; $K = 100$; $r = 5$ % ; $s_1 = s_2 = 25$ % ; $r = .$; $t = 1$ year .

figure 14

The used values of the parameters imply that when the correlation is perfect, the duration of this option is worth the one of an American put option on a single risky asset.

As it has been done for single option, the algorithm can be directly generalized to outside barrier options and other pay off structures with non-constant diffusion parameters.

Conclusion

In this paper, a new method providing the estimation of the lifetime distributions of the most common American and exotic options has been presented using a tree. Of course, more general trees than the one of CRR can be used in order to obtain this distribution without changing the algorithm and the application interest rate options is the same. One of the great advantages of our methods is that we can deal with non-constant parameters. Trends structure, volatility smile, dividend and even several stochastic characteristics can be taken into account like it has been done for option contingent to two underlying assets. But the high sensibility— referring to the parameters of the option — of the value of the duration has come to light with the developed results. This implies either a perfect knowledge of the values employed or a study of the sensibility of the lifetime distribution to those parameters.

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[1] A nearly complete description of exotic options can been found in Nelken [11] for example.

[2] In the risk neutral world, the diffusion process of the risky asset S is assumed to be the following geometric Brownian motion : $\frac{dS}{S} = rdt + s\ dZ$ where r is the risk free rate, σ the volatility of the return of S and Z a standard Brownian motion.

[3] As the last period is not concerned with anticipated exercise, only the first two periods are studied.

[4] For American call and put options there is only one value per period of the EndProb matrix that is positive. But for some exotic options, as double barrier one, two values of EndProb matrix can be positive for a given period.

[5] 1500 time steps have been used to estimate this distribution. This calculation doesn't last more than a second on a pentium II 350 processor.

[6] In the real world American call options can be prematurely exercised since there are many transactions costs.

[7] A complete study of the problem encountered with pricing barrier option with the CRR's binomial tree is provided in Boyle & Lau's articles.

[8] Probabilities of crossing a barrier are correct only in discrete time. But their values are not correct in continuous time.

[9] In this section, the model of Augros & Moreno has been employed. In continuous time and in the risk neutral world, the processes of the 2 risky assets are geometric Brownian motions: $\frac{dS_i}{S_i} = rdt + s_i dZ_i$ for i = 1, 2 with $dZ_1 dZ_2 = r\, dt$.

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